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Flow Turning in Solid-Propellant Rocket Combustion Stability Analyses

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Culick developed the concept of a "flow-turning" effect from a comparison of the one- and three-dimensional models of acoustic stability of combustion in a solid-propellant rocket motor. Several terms occur in the one-dimensional model which are absent from the three-dimensional model. Culick suggested adding these terms to the three-dimensional model in an ad hoc manner, arguing that they represent physical phenomena which are not accounted for in the analysis. A procedure is developed here to allow comparison of the one- and three-dimensional models by averaging over the cross section of the motor activity. This procedure results in the missing terms arising naturally as a result of the inclusion of the three-dimensional boundary conditions in the averaging process. This result raises serious questions concerning the ad hoc inclusion of terms into the three-dimensional stability analysis without a clear physical basis.

Nomenclature

A = cross-sectional area
a = sound speed
e = total specific energy
h = total specific enthalpy
ℓ = unit vector in the plane of the cross section
M = Mach number
m = mass generation per unit area
n = unit vector in three-dimensional space
P = pressure
Q = arbitrary scalar variable
s = distance along perimeter of cross section
T = temperature
t = time
u = specific internal energy
V = velocity
x = Cartesian coordinate
ϵ = measure of strength of acoustic waves
γ = ratio of specific heats
ρ = density
σ = surface area of the cavity

Subscripts

i,j = component of a Cartesian tensor
s = evaluated on the surface

Introduction

SOLID-PROPELLANT rocket motors frequently exhibit an unsteady behavior during their operation. The frequency of the observed oscillations can often be identified with a classical acoustical mode of the motor cavity.[1] Thus, questions concerning the combustion stability of a motor are often equivalent to determining whether the acoustic modes grow or decay. Two different theoretical models have been developed to predict the stability of the acoustic oscillations in a motor. The first is due to Hart and McClure[2] and the second by Culick,[3-6] Flandro,[7] and others. The Hart and McClure model deals directly with an energy balance of the combustion-generated gas. It fails to differentiate between acoustic disturbances and other (rotational) disturbances which can occur. The theory appears to answer the question of whether the energy of all possible disturbances is increasing rather than if the energy in the acoustic disturbances alone is increasing.

The second theory, mainly identified with Culick, begins with the full inviscid equations of fluid mechanics, and develops the equations governing acoustic oscillations and the interaction of these acoustic disturbances with the mean flow and boundaries of the cavity. From these results the rate of growth or decay of the acoustic waves can be determined. As in the first model, stability is determined by whether the energy in the disturbance is increasing or decreasing, but in this case only the acoustic disturbance is considered. Since terms are present which both increase and decrease the energy in the acoustic disturbances it is vital that all of the sources and sinks of energy be identified and modeled as carefully as possible to accurately predict the stability boundary.

Beginning in 1972 Culick[4-6] described a comparison of the one- and the three-dimensional stability analyses and noted that the three-dimensional analysis, when reduced to one dimension, lacked terms present in the purely one-dimensional model. Culick then argues that this apparent inconsistency can be removed by adding terms into the three-dimensional model in an ad hoc manner (a process he calls patching). He then argues that the missing terms represent viscous processes implicit in the one-dimensional formulation of the problem but missing from the three-dimensional formulation. These terms have to do with the turning of the flow from a direction normal to the cavity boundary to the direction in which the acoustic waves are moving and have become known as flow turning.

This paper discusses the appropriateness of this patching procedure but does this by beginning at a very fundamental level, asking what is the relationship between one- and three-dimensional models in fluid mechanics. It is a well-known dogma of fluid mechanics that one-dimensional models give the "average" values of the flow variables. This concept has been discussed briefly by Shapiro,[8] Crocco,[9] and Zucrow and Hoffman.[10] Clemins[11] described the errors that are made as a result of assuming that the average of the product of terms is the product of the average, as is usually done.

The usual procedures to obtain one-dimensional equations are either to neglect derivatives and velocity components in the three-dimensional equations perpendicular to the single one-dimensional spatial coordinate or to rederive the equations using an infinitesimal control volume with variation allowed only in a single spatial direction. The equations obtained by the first and the second methods will

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agree, provided there is no flow entering the region of interest perpendicular to the one-dimensional coordinate. If such a flow exists, only the second method yields the correct equations which incorporate the entering flow as inhomogeneous source terms.

A third method is developed here. The procedure is to formally average the three-dimensional equation over the region of interest in planes perpendicular to the one-dimensional coordinate. This averaging procedure requires that the three-dimensional boundary conditions be applied and gives rise to the inhomogeneous source terms in the governing equations in a natural way.

The Averaging Procedure

Consider the geometric configuration shown in Fig. 1. If the x_1 coordinate is chosen as the single one-dimensional spatial coordinate, the average of some flow property Q is given in the usual manner by

$$\langle Q\rangle = \frac{1}{A}\int_A Q\mathrm{d}A \tag{1}$$

where the element of area dA lies in a plane parallel to the x_2-x_3 plane. If Q is a function of three space variables and time, the resulting average is a function of x_1 and time only. The cross-sectional area A may be a function of x_1 but not of time.

In attempting to apply this averaging procedure to either the flow or acoustic equations, four general types of averages occur. The first of these is

$$\left\langle\frac{\partial Q}{\partial t}\right\rangle = \frac{1}{A}\int_A \frac{\partial Q}{\partial t}\mathrm{d}A \tag{2}$$

and since area is not a function of time this is easily seen to be

$$\left\langle\frac{\partial Q}{\partial t}\right\rangle = \frac{\partial}{\partial t}\left(\frac{1}{A}\int_A Q\mathrm{d}A\right) = \frac{\partial\langle Q\rangle}{\partial t} \tag{3}$$

The second type of term is given by

$$\left\langle\frac{\partial}{\partial x_j}(QV_j)\right\rangle = \frac{1}{A}\int_A \frac{\partial}{\partial x_j}(QV_j)\mathrm{d}A \quad [j=2 \text{ to } 3] \tag{4}$$

Cartesian tensor notation is used rather than vector notation as this will be particularly useful below. Since some results are applicable only in the x_2-x_3 plane the notation [$j=2$ to 3] will be given to minimize confusion. The integral given here is a two-dimensional form of Gauss' theorem[12] and can be expressed as

$$\int\frac{\partial}{\partial x_j}(QV_j)\mathrm{d}A = \oint QV_j\ell_j\mathrm{d}s \quad [j=2 \text{ to } 3] \tag{5}$$

where ℓ_j is a component of the unit normal to the curve bounding the area A and lies in the plane of integration. Distance along this bounding curve is given by s. Thus

$$\left\langle\frac{\partial}{\partial x_j}(QV_j)\right\rangle = \frac{1}{A}\oint QV_j\ell_j\mathrm{d}s \quad [j=2 \text{ to } 3] \tag{6}$$

A term of the form

$$\left\langle\frac{\partial Q}{\partial x_j}\right\rangle = \frac{1}{A}\int_A \frac{\partial Q}{\partial x_j}\mathrm{d}A \quad [j=2 \text{ or } 3] \tag{7}$$

is the third type of integral. This integral is easily shown by use of Green's theorem to be

$$\left\langle\frac{\partial Q}{\partial x_j}\right\rangle = \frac{1}{A}\oint Q\ell_j\mathrm{d}s \quad [j=2 \text{ or } 3] \tag{8}$$

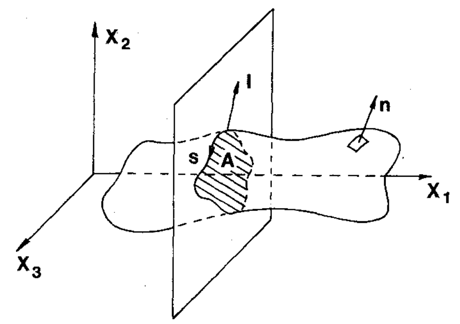

Fig. 1 Geometric configuration used for the averaging process.

The fourth type of average

$$\left\langle\frac{\partial Q}{\partial x_1}\right\rangle = \frac{1}{A}\int_A \frac{\partial Q}{\partial x_1}\mathrm{d}A \tag{9}$$

is somewhat more involved and will be discussed in terms of the geometry shown in Fig. 2. The integral in Eq. (9) is given by

$$\int_{a(x_1)}^{b(x_1)}\int_{g(x_1,x_2)}^{f(x_1,x_2)} \frac{\partial Q(x_1,x_2,x_3,t)}{\partial x_1}\mathrm{d}x_3\mathrm{d}x_2 \tag{10}$$

and using the Leibnitz rule twice and noting that

$$A = \int_a^b\int_g^f \mathrm{d}x_3\mathrm{d}x_2 \tag{11}$$

Eq. (9) can be written as

$$\left\langle\frac{\partial Q}{\partial x_1}\right\rangle = \frac{1}{A}\frac{\partial}{\partial x_1}(A\langle Q\rangle) - \frac{1}{A}\oint Q\frac{\partial^2 A}{\partial x_1\partial s}\mathrm{d}s \tag{12}$$

However, a second form is also useful. If a normal to the bounding surface of the region of interest is given by a unit vector n with components n_j [$j=1$ to 3] then the term

$$\frac{\partial^2 A}{\partial s\partial x_1}$$

can be expressed in terms of the components of n giving

$$\left\langle\frac{\partial Q}{\partial x_1}\right\rangle = \frac{1}{A}\frac{\partial}{\partial x_1}(A\langle Q\rangle) + \frac{1}{A}\oint Q\frac{n_1\mathrm{d}s}{\sqrt{n_2^2+n_3^2}} \tag{13}$$

A final result can be obtained by combining Eqs. (6) and (13) and noting that

$$\ell_j = n_j/\sqrt{n_2^2+n_3^2} \quad [j=2 \text{ to } 3] \tag{14}$$

to give

$$\left\langle\frac{\partial}{\partial x_j}(QV_j)\right\rangle \quad [j=1 \text{ to } 3]$$

$$= \left\langle\frac{\partial}{\partial x_j}(QV_j)\right\rangle + \left\langle\frac{\partial}{\partial x_1}(QV_1)\right\rangle \quad [j=2 \text{ to } 3]$$

$$= \frac{1}{A}\frac{\partial}{\partial x_1}(A\langle QV_1\rangle) + \frac{1}{A}\oint QV_j\frac{n_j}{\sqrt{n_2^2+n_3^2}}\mathrm{d}s \quad [j=1 \text{ to } 3] \tag{15}$$

where Q in Eq. (13) has been replaced by QV_1.

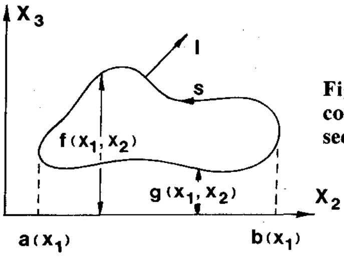

Fig. 2 The geometric configuration of a cross section of the motor cavity.

As an example of the use of Eqs. (2), (8), and (15), consider their application to the three-dimensional equations of fluid mechanics as given by

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho V_j)=0 \tag{16}$$

$$\frac{\partial(\rho V_i)}{\partial t}+\frac{\partial}{\partial x_j}(\rho V_i V_j)+\frac{\partial P}{\partial x_i}=0 \tag{17}$$

and

$$\frac{\partial(\rho e)}{\partial t}+\frac{\partial}{\partial x_j}(\rho V_j e)+\frac{\partial}{\partial x_j}(P V_j)=0 \quad [j \text{ and } i=1 \text{ to } 3] \tag{18}$$

with the boundary condition

$$\rho V_i n_i = -m \qquad [i=1 \text{ to } 3] \tag{19}$$

on the surface of the region in which the flow occurs. The resulting equations are

$$A\frac{\partial\langle\rho\rangle}{\partial t}+\frac{\partial}{\partial x_1}(A\langle\rho V_1\rangle)=\oint\frac{m\,ds}{\sqrt{n_2^2+n_3^2}} \tag{20}$$

$$A\frac{\partial\langle\rho V_1\rangle}{\partial t}+\frac{\partial}{\partial x_1}(A\langle\rho V_1^2\rangle)+\frac{\partial}{\partial x_1}(A\langle p\rangle) - \oint P\frac{\partial^2 A}{\partial s\partial x_1}ds=\oint V_1\frac{m}{\sqrt{n_2^2+n_3^2}}ds \tag{21}$$

where $i=1$ in Eq. (17),

$$\frac{\partial}{\partial t}(A\langle\rho V_i\rangle)+\frac{\partial}{\partial x_1}(A\langle\rho V_1 V_i\rangle)+\oint P\ell_i ds=\oint V_i m\frac{ds}{\sqrt{n_2^2+n_3^2}} \tag{22}$$

when $i=2$ or 3 in Eq. (17), and

$$\frac{\partial\langle\rho e\rangle}{\partial t}-\frac{\partial}{\partial x_1}(A\langle\rho V_1 e\rangle)+\frac{\partial}{\partial x_1}(A\langle P V_1\rangle) = \oint\frac{e m\,ds}{\sqrt{n_2^2+n_3^2}}+\oint\frac{P}{\rho}\frac{m\,ds}{\sqrt{n_2^2+n_3^2}} \tag{23}$$

If the pressure in the fourth term in Eq. (21) is assumed to be $\langle P\rangle$, as it usually is in a one-dimensional model, then the third and fourth terms combine to give $A\partial\langle P\rangle/\partial x_1$ and Eqs. (19-23) are the usual equations of one-dimensional flow with sources except for the factor $(n_2^2+n_3^2)^{-1/2}$. This factor is the reciprocal of the cosine of the divergence between the x_1 axis and the region boundary. One-dimensional flow is understood to be useful for slowly diverging ducts or when this factor is approximately one.

The Acoustic Stability Equations

The derivation of the acoustic equations governing the stability problem is straightforward but rather long and laborious. Thus the derivation will be carried out in detail for the three-dimensional case, but only the final results for the one-dimensional case will be given. The derivation of the one-dimensional results follows in a manner exactly parallel to the three-dimensional case.

The procedure used here follows that of Flandro[7] but, to simplify the averaging process which will be carried out later, the equations are maintained in a divergence form. The basic approach is to expand the equations of inviscid fluid mechanics (16-18) and the boundary conditions (19) in terms of two small parameters, the Mach number M of the flow as it enters the cavity due to the combustion process and ϵ which is a measure of the strength of the acoustic waves in the motor. All of the flow variables are first expanded in a power series in ϵ, for example, pressure in the cavity will be written as

$$P=P^{(0)}+P^{(1)}\epsilon+P^{(2)}\epsilon^2+\ldots \tag{24}$$

where $P^{(0)}$ represents the pressure in the motor in the absence of acoustic waves, $P^{(1)}$ represents the linear acoustic waves, etc. Carrying out this process and separating the resulting equations by the powers of ϵ yield for ϵ^0

$$\frac{\partial\rho^{(0)}}{\partial t}+\frac{\partial}{\partial x_j}(\rho^{(0)}V_j^{(0)})=0 \tag{25}$$

$$\frac{\partial(\rho^{(0)}V_i^{(0)})}{\partial t}+\frac{\partial}{\partial x_j}(\rho^{(0)}V_i^{(0)}V_j^{(0)})+\frac{\partial P^{(0)}}{\partial x_i}=0 \tag{26}$$

and

$$\frac{\partial(\rho^{(0)}e^{(0)})}{\partial t}+\frac{\partial}{\partial x_j}(\rho^{(0)}e^{(0)}V_j^{(0)})+\frac{\partial}{\partial x_j}(\rho^{(0)}V_j^{(0)})=0 \tag{27}$$

with the boundary condition

$$\rho^{(0)}V_j^{(0)}n_j=-m^{(0)} \tag{28}$$

These equations govern the mean flow and will be assumed to be steady as is usually done.[3,7]

The equations governing the linear disturbance are of order ϵ^1 and are given by

$$\frac{\partial(\rho^{(1)})}{\partial t}+\frac{\partial}{\partial x_j}(\rho^{(1)}V_j^{(0)}+\rho^{(0)}V_j^{(1)})=0 \tag{29}$$

$$\frac{\partial}{\partial t}(\rho^{(1)}V_i^{(0)}+\rho^{(0)}V_i^{(1)})+\frac{\partial}{\partial x_j}(\rho^{(1)}V_j^{(0)}V_i^{(0)} +\rho^{(0)}V_j^{(1)}V_i^{(0)}+\rho^{(0)}V_j^{(0)}V_i^{(1)})+\frac{\partial P^{(1)}}{\partial x_i}=0 \tag{30}$$

$$\frac{\partial}{\partial t}(\rho^{(0)}e^{(1)}+\rho^{(1)}e^{(0)})+\frac{\partial}{\partial x_j}(\rho^{(1)}e^{(0)}V_j^{(0)}+\rho^{(0)}e^{(1)}V_j^{(0)} +\rho^{(0)}e^{(0)}V_j^{(1)})+\frac{\partial}{\partial x_j}(P^{(1)}V_j^{(0)}+P^{(0)}V_j^{(1)})=0 \tag{31}$$

and the boundary condition

$$\rho^{(1)}V_j^{(0)}n_j+\rho^{(0)}V_j^{(1)}n_j=-m^{(1)} \tag{32}$$

In addition to the above equations, an ideal gas will be assumed to give the required equations of state.

An inhomogeneous wave equation can be obtained from Eq. (30) and (31) and the equations of state, yielding

$$\frac{\partial^2 P^{(1)}}{\partial x_i^2} - \frac{1}{(a^{(0)})^2}\frac{\partial^2 P^{(1)}}{\partial x_i^2} = -\frac{\partial^2}{\partial x_i \partial x_j}(\rho^{(1)} V_j^{(0)} V_i^{(0)} + \rho^{(0)} V_j^{(1)} V_i^{(0)} + \rho^{(0)} V_j^{(0)} V_i^{(1)}) - \frac{\partial^2}{\partial x_i \partial t}(\rho^{(1)} V_i^{(0)}$$

$$+\rho^{(0)} V_i^{(1)}) + \frac{(\gamma-1)}{(a^{(0)})^2}\frac{\partial^2}{\partial t^2}\left(\rho^{(0)} V_i^{(0)} V_i^{(1)} + \frac{1}{2}\rho^{(1)} V_i^{(0)} V_i^{(0)}\right) + \frac{1}{(a^{(0)})^2}\frac{\partial^2}{\partial t \partial x_j}\left[\gamma P^{(1)} V_j^{(0)} + \gamma P^{(0)} V_j^{(1)}\right.$$

$$\left. + \frac{(\gamma-1)}{2}(\rho^{(1)} V_i^{(0)} V_i^{(0)} V_j^{(0)} + 2\rho^{(0)} V_i^{(0)} V_i^{(1)} V_j^{(0)} + \rho^{(0)} V_i^{(0)} V_i^{(0)} V_j^{(1)})\right] \tag{33}$$

Each term in the initial expansions, of which Eq. (24) is an example, is now expanded in terms of the Mach number M, such that, for example,

$$P^{(0)} = P^{(0,0)} + P^{(0,1)} M + \ldots \tag{34}$$

and

$$P^{(1)} = P^{(1,0)} + P^{(1,1)} M + \ldots \tag{35}$$

Making this substitution into the mean flow equations (25-28) and assuming the mean flow is steady requires $m^{(0,0)} = 0$ since if the Mach number of the flow entering the cavity is zero, no flow is entering. The velocity $V^{(0,0)}$ is assumed zero and this implies $P^{(0,0)}$ is constant. $\rho^{(0,0)}$ and $T^{(0,0)}$ are also assumed constant. The first-order mean flow velocity $V^{(0,1)}$ is found to be nonzero, but the rest of the first-order mean flow variables can be taken to be zero with no loss of generality.

Making the expansions in the wave equation (33) and using the results discussed above gives

$$\frac{\partial^2 P'}{\partial x_i^2} - \frac{1}{(a^{(0,0)})^2}\frac{\partial^2 P'}{\partial t^2} = M\left\{-\frac{\partial}{\partial x_i}\frac{\partial}{\partial t}(\rho^{(1,0)} V_i^{(0,1)}) - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}[\rho^{(0,0)}(V_i^{(0,1)} V_j^{(1,0)}\right.$$

$$\left. + V_i^{(1,0)} V_j^{(0,1)})] + \frac{\gamma-1}{(a^{(0,0)})^2}\rho^{(0,0)}\frac{\partial^2}{\partial t^2}(V_j^{(0,1)} V_j^{(1,0)}) + \frac{1}{(a^{(0,0)})^2}\frac{\partial}{\partial t}\frac{\partial}{\partial x_j}(\gamma P^{(1,0)} V_j^{(0,1)})\right\} \tag{36}$$

where $P' = P^{(1,0)} + P^{(1,1)} M$. Equation (36) is the equivalent of the wave equation obtained by Culick[6] and can be put in his form by expanding the derivatives and substituting from Eqs. (25-27) and (29-31).

If the one-dimensional equations are used and a parallel procedure is followed the resulting equation is

$$\frac{\partial}{\partial x_1}\left(A\frac{\partial P'}{\partial x_1}\right) - \frac{1}{(a^{(0,0)})^2}\frac{\partial^2 (P'A)}{\partial t^2} = M\left\{-\frac{\partial^2}{\partial x_1,\partial t}(\rho^{(1,0)} V_1^{(0,1)} A) - \frac{\partial^2}{\partial x_1^2}(2\rho^{(0,0)} V_1^{(0,1)} V_1^{(1,0)} A)\right.$$

$$+\frac{\gamma-1}{(a^{(0,0)})^2}\rho^{(0,0)}\frac{\partial^2}{\partial t^2}(V_1^{(0,1)} V_1^{(1,0)} A) + \frac{1}{(a^{(0,0)})^2}\frac{\partial}{\partial t}\frac{\partial}{\partial x_1}(\gamma P^{(1,0)} V_1^{(0,1)} A) + \frac{\partial}{\partial x_1}\left(\oint u^{(1,0)} m^{(0,1)} \mathrm{d}s\right)$$

$$\left. -\frac{(\gamma-1)}{(a^{(0,0)})^2}\frac{\partial}{\partial t}\left[\oint (m^{(1,1)} h^{(0,0)} + m^{(0,1)} h^{(1,0)})\mathrm{d}s\right]\right\} \tag{37}$$

The averaging procedure developed above can now be applied to the inhomogeneous wave equation (36) yielding

$$\frac{\partial^2}{\partial x_1^2}(A\langle P'\rangle) - \frac{\partial}{\partial x_1}\oint P'\frac{\partial^2 A}{\partial x_1 \partial s}\mathrm{d}s - \frac{1}{(a^{(0,0)})^2}\frac{\partial^2}{\partial t^2}(A\langle P'\rangle) = -\frac{\partial^2}{\partial x_1 \partial t}(A\langle \rho^{(1,0)} V_1^{(0,1)}\rangle)$$

$$-\frac{\partial^2}{\partial x_1^2}(2A\langle \rho^{(0,0)} V_1^{(0,1)} V_1^{(1,0)}\rangle) + \frac{\gamma-1}{(a^{(0,0)})^2}\rho^{(0,0)}\frac{\partial^2}{\partial t^2}(A\langle V_i^{(0,1)} V_i^{(1,0)}\rangle) + \frac{\gamma}{(a^{(0,0)})^2}\frac{\partial^2}{\partial x_1 \partial t}(A\langle P^{(1,0)} V_1^{(0,1)}\rangle)$$

$$-\oint \frac{\partial}{\partial x_i}(P^{(1,1)}) n_i \frac{\mathrm{d}s}{\sqrt{n_2^2+n_3^2}} - \frac{\partial}{\partial t}\oint \rho^{(1,0)} V_i^{(0,1)} n_i \frac{\mathrm{d}s}{\sqrt{n_2^2+n_3^2}} - \frac{\partial}{\partial x_1}\oint \rho^{(0,0)}(V_i^{(0,1)} V_1^{(1,0)}$$

$$+ V_i^{(1,0)} V_1^{(0,1)}) n_i \frac{\mathrm{d}s}{\sqrt{n_2^2+n_3^2}} - \oint \frac{\partial}{\partial x_i}[\rho^{(0,0)}(V_i^{(0,1)} V_j^{(1,0)} + V_i^{(1,0)} V_j^{(0,1)})] n_j \frac{\mathrm{d}s}{\sqrt{n_2^2+n_3^2}}$$

$$+\frac{\gamma}{(a^{(0,0)})^2}\frac{\partial}{\partial t}\oint P^{(1,0)} V_i^{(0,1)} n_i \frac{\mathrm{d}s}{\sqrt{n_2^2+n_3^2}} \tag{38}$$

It is straightforward to show that the first and fourth integrals on the right-hand side of Eq. (38) give

$$-\frac{\partial}{\partial t}\oint m^{(1,1)}\frac{\mathrm{d}s}{\sqrt{n_2^2+n_3^2}} - \frac{\gamma-1}{(a^{(0,0)})^2}\frac{\partial}{\partial t}\oint m^{(1,1)} h^{(0,0)}\frac{\mathrm{d}s}{\sqrt{n_2^2+n_3^2}} \tag{39}$$

after substitution from the expanded forms of Eqs. (28) and (32). Similarly the second and fifth integrals give

$$\frac{\partial}{\partial t}\oint\frac{\rho^{(1,0)}m^{(0,1)}}{\rho^{(0,0)}}\frac{ds}{\sqrt{n_2^2+n_3^2}}-\frac{\partial}{\partial t}\oint\frac{P^{(1,0)}m^{(0,1)}}{P^{(0,0)}}\frac{ds}{\sqrt{n_2^2+n_3^2}}=-\frac{\gamma-1}{(a^{(0,0)})^2}\frac{\partial}{\partial t}\oint m^{(0,1)}h^{(1,0)}\frac{ds}{\sqrt{n_2^2+n_3^2}} \tag{40}$$

and the third integral can be written as

$$\frac{\partial}{\partial x_1}\oint\frac{m^{(0,1)}V_1{}^{(1,0)}ds}{\sqrt{n_2^2+n_3^2}} \tag{41}$$

The averaged form of the wave equation then becomes

$$\frac{\partial^2}{\partial x_1^2}(A\langle P'\rangle)-\frac{\partial}{\partial x_1}\oint P'\frac{\partial^2 A}{\partial x_1\partial s}ds-\frac{1}{(a^{(0,0)})^2}\frac{\partial^2}{\partial t^2}(A\langle P'\rangle)=-\frac{\partial^2}{\partial x_1\partial t}(A\langle\rho^{(1,0)}V_1{}^{(0,1)}\rangle)-\frac{\partial^2}{\partial x_1^2}(2A\langle\rho^{(0,0)}V_1{}^{(0,1)}V_1{}^{(1,0)}\rangle)$$
$$+\frac{\gamma-1}{(a^{(0,0)})^2}\rho^{(0,0)}\frac{\partial^2}{\partial t^2}(A\langle V_i{}^{(0,1)}V_i{}^{(1,0)}\rangle)+\frac{\gamma}{(a^{(0,0)})^2}\frac{\partial^2}{\partial x_1\partial t}(A\langle P^{(1,0)}V_1{}^{(0,1)}\rangle)+\frac{\partial}{\partial x_1}\oint\frac{m^{(0,1)}V_1{}^{(1,0)}ds}{\sqrt{n_2^2+n_3^2}}$$
$$-\frac{\gamma-1}{(a^{(0,0)})^2}\frac{\partial}{\partial t}\oint(m^{(1,1)}h^{(0,0)}+m^{(0,1)}h^{(1,0)})\frac{ds}{\sqrt{n_2^2+n_3^2}} \tag{42}$$

Equation (42) is identical to the one-dimensional result given by Eq. (37), if the value of P on the boundary is equal to the average value over the cross section and

$$\langle V_i{}^{(0,1)}V_i{}^{(1,0)}\rangle=\langle V_1{}^{(0,1)}V_1{}^{(1,0)}\rangle \tag{43}$$

which is quite reasonable if the flow is predominated along the x axis and if the cavity is only slightly divergent or convergent so that

$$\sqrt{n_2^2+n_3^2}=1 \tag{44}$$

Discussion

It has been shown above that the one- and three-dimensional inviscid formulations and the combustion stability problem contain the same physical phenomena. Yet the one-dimensional stability model yields terms not present in the three-dimensional model.[6] The origin of these terms is best considered in inspecting the equation for the complex-wave number. The derivation of this equation follows exactly the approach of Culick.[5,6] In the present notation and with assumptions made earlier this equation is

$$(k^2-k_e^2)E_e^2=i\rho^{(0,0)}(a^{(0,0)})^2k_eM\Big[\hat P^{(1,0)}\Big(\hat u^{(1,1)}+\frac{u^{(0,1)}\hat P^{(1,0)}}{\rho^{(0,0)}(a^{(0,0)})^2}\Big)A-\int_0^L\hat P^{(1,0)}\oint\frac{\hat m^{(1,1)}}{\rho^{00}}dsdx_1\Big]$$
$$+i\frac{k_e}{a^{(0,0)}}\Big[\frac{1}{\rho^{(0,0)}}\int_0^L\frac{1}{k_e^2}\Big(\frac{\partial\hat P^{(1,0)}}{\partial x_1}\Big)^2\oint m^{(0,1)}dsdx_1-\int_0^L A\Big(\frac{1}{A}\oint\hat u_s{}^{(1,0)}m^{(0,1)}ds\Big)\frac{\partial\hat P^{(1,0)}}{\partial x_1}dx_1\Big] \tag{45}$$

for the one-dimensional model and

$$(k-k_e^2)E_e^2=-i\rho^{(0,0)}(a^{(0,0)})^2k_e\int_\sigma P^{(1,0)}\Big(\hat u_i{}^{(1,1)}+\frac{u_i{}^{(0,1)}\hat P^{(1,0)}}{\rho^{(0,0)}(a^{(0,0)})^2}\Big)n_i d\sigma \tag{46}$$

for the three-dimensional model. Symbols with a caret are the remaining spatial variation of a variable after the time dependence, $\exp(i(k/a^{(0,0)})t)$, has been removed, e.g.,

$$P^{(1,0)}=\hat P^{(1,0)}\exp(i(k/a^{(0,0)})t) \tag{47}$$

These equations are the exact equivalent of Eqs. (3.9) and (2.29) in Culick's[6] comparison if particles, residual combustion, and temperature differences between the injected flow and the flow in the bulk of the chamber are ignored. The subscript s in the last term of Eq. (45) indicates that the velocity $u^{(1,0)}$ is evaluated on the cavity surface. Since

$$m^{(1,1)}=(\rho^{(0,0)}u_i{}^{(1,1)}+\rho^{(1,0)}u_i{}^{(0,1)})n_i \tag{48}$$

and

$$\hat\rho^{(1,0)}=\frac{\hat P^{(1,0)}}{(a^{(0,0)})^2} \tag{49}$$

the first two terms in Eq. (45) are the equivalent of the surface integral in Eq. (46). Culick,[6] of course, recognizes this fact and discusses it in connection with his Eq. (3.11). He also recognizes that the remaining two terms in Eq. (45) would cancel if $u_s^{(1,0)} = u^{(1,0)}$ as a result of the linearized momentum equation, which can be obtained from Eq. (30). Culick[6] (p. 117), however, chooses to assume $u_s^{(1,0)}$ to be zero for "... flow coming into the air at the burning surface, because there is not parallel motion." This assumption then eliminates one term of the pair that could cancel and leads to the flow-turning term. Two criticisms of the assumption that $u_s^{(1,0)}$ is zero can be made.

The first criticism is one of formal mathematics. Within the formal perturbation schemes used here and implied in Culick's work the velocity $u^{(1,0)}$ at any point in the cavity is determined from a solution for the pressure. The pressure in turn is governed by the usual homogeneous wave equation, in the inviscid case, and the boundary condition that the gradient of pressure normal to a solid boundary is zero. The parallel component of velocity will, in general, not be zero, and any attempt to specify it as such results in an overspecification of the problem. The component of velocity $u_s^{(1,0)}$ is not the component of velocity resulting from injection or burning at the surface. This mass addition $m^{(1,1)}$ is of the next higher order in M.

On physical grounds, the assumption of $u_s^{(1,0)}$ being zero would appear to be an attempt to model a viscous phenomenon, the acoustic boundary layer with mass injection. A correct model of this phenomenon would require including the viscous terms in the three-dimensional equations of fluid mechanics and carrying out a boundary-layer type of analysis. A preliminary analysis of this type has been carried out by Flandro.[13] The model as proposed by Culick has not been developed in a mathematically rigorous manner. It is clear, however, that his model does yield correct general behavior, but the lack of a formal physical and mathematical background raises serious questions concerning its ad hoc addition into the stability analysis.

Conclusions

The terms that produce the flow-turning effect arise naturally in converting the three-dimensional result into a one-dimensional result by the averaging process. An essential feature of this technique is that the boundary conditions are utilized, while in other comparisons they are ignored.

The results obtained here indicate that the one-dimensional model does not contain any physical phenomena missing from the three-dimensional model and raises serious questions concerning the ad hoc addition of any terms into the three-dimensional model.

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